JOURNAL OF APPROXIMATION THEORY 71, 121–122 (1992)

Note

On the Maximum Principle of Bernstein Polynomials on a Simplex

THOMAS SAUER

Mathematical Institute, University of Erlangen–Nuremberg, Bismarckstrasse $1\frac{1}{2}$, 8520 Erlangen, Germany

Communicated by A. Pinkus

Received June 7, 1991; revised July 25, 1991

In a recent paper [1] Chang and Zhang gave a proof of what they called a *converse theorem of convexity.* Their bivariate result was extended to a multivariate (and tensor product) setting by Dahmen and Micchelli [2] using semigroup techniques. In fact, the latter proved among other things:

THEOREM. *Suppose f is a continuous function on S_m such that for the Bernstein polynomials $B_n f$ the following holds:*

$$B_n f(u) \geqslant f(u), \qquad u \in S_m, n \in \mathbb{N}.$$

Then f achieves its maximum on ∂S_m, the boundary of S_m.

To make the theorem understandable let me briefly recall the definitions. Since a point of any m-dimensional simplex can be identified with its barycentric coordinates we can always consider the simplex to be the "barycentric unit simplex" $S_m = \{u = (u_0, ..., u_m) : u_k \geqslant 0, \sum_{k=0}^m u_k = 1\}$. The *Bernstein–Bézier basis polynomials* B_i^n: $S_m \to \mathbb{R}$ are then defined via

$$B_i^n(u) = \frac{n!}{i!} u^i = \frac{n!}{i_0! \cdots i_m!} u_0^{i_0} \cdots u_m^{i_m},$$

where $i = (i_0, ..., i_m)$ is a multiindex with $|i| = \sum_{k=0}^m i_k = n$. For any $f \in C(S_m)$, the nth *Bernstein polynomial* $B_n f$ is given by

$$B_n f(u) = \sum_{|i| = n} f\left(\frac{i}{n}\right) B_i^n(u).$$

I want to give a short and elementary proof of the theorem stated above; in fact, it is an immediate consequence of

0021-9045/92 $5.00

LEMMA. *Suppose $f \in C(S_m)$ is such that $B_n f \geqslant f$ for all $n \in \mathbb{N}$. If f achieves its maximum inside S_m, then f is constant.*

Proof. Suppose f achieves its maximum at some u^0 inside S_m. This means that $u_k^0 > 0$, $k = 0, ..., m$, and hence $B_i^n(u^0) > 0$, $|i| = n$. Since $\sum_{|i|=n} B_i^n(u) \equiv 1$ we have, due to the assumptions,

$$B_n f(u^0) \geqslant f(u^0) = f(u^0) \sum_{|i|=n} B_i^n(u^0) \geqslant \sum_{|i|=n} f\left(\frac{i}{n}\right) B_i^n(u^0) = B_n f(u^0),$$

with equality if and only if $f(u^0) = f(i/n)$, $|i| = n, n \in \mathbb{N}$. Since f is continuous, it indeed must be a constant. ∎

Let us recall that the restriction of $B_n f$ to an $(m-1)$-dimensional face of S_m is the same as the $(m-1)$-variate Bernstein polynomial of the restriction of f to this face. An application of a lower-dimensional version of the theorem shows that the maximum has to be located on the boundary of the face. By iteration we finally obtain

COROLLARY. *Suppose $f \in C(S_m)$ such that $B_n f \geqslant f$. Then f achieves its maximum at one of the vertices of S_m.*

REFERENCES

1. G. CHANG AND J. ZHANG, Converse theorems of convexity for Bernstein polynomials over triangles, *J. Approx. Theory* **61** (1990), 265–278.
2. W. DAHMEN AND C. A. MICCHELLI, Convexity and Bernstein polynomials on k-simploids, *Acta Math. Appl. Sinica* **6** (1990), 50–66.

Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis